Communication Engineering Systems

LTI System and Filtering (3)

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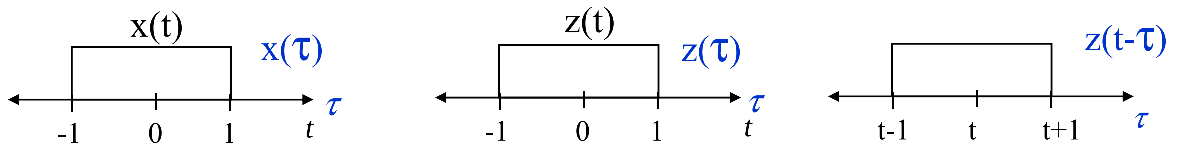
"All things are difficult before they are easy"

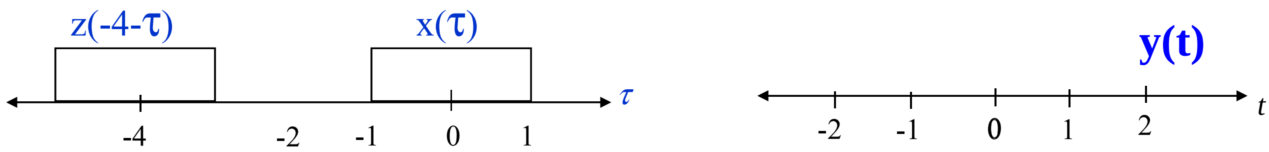
Outline

- ❑ Review of Convolution
- ❑ Special Functions
- ❑ Sampling
- ❑ Linear Time-Invariant (LTI) System
- ❑ Filtering
- ❑ Correlation

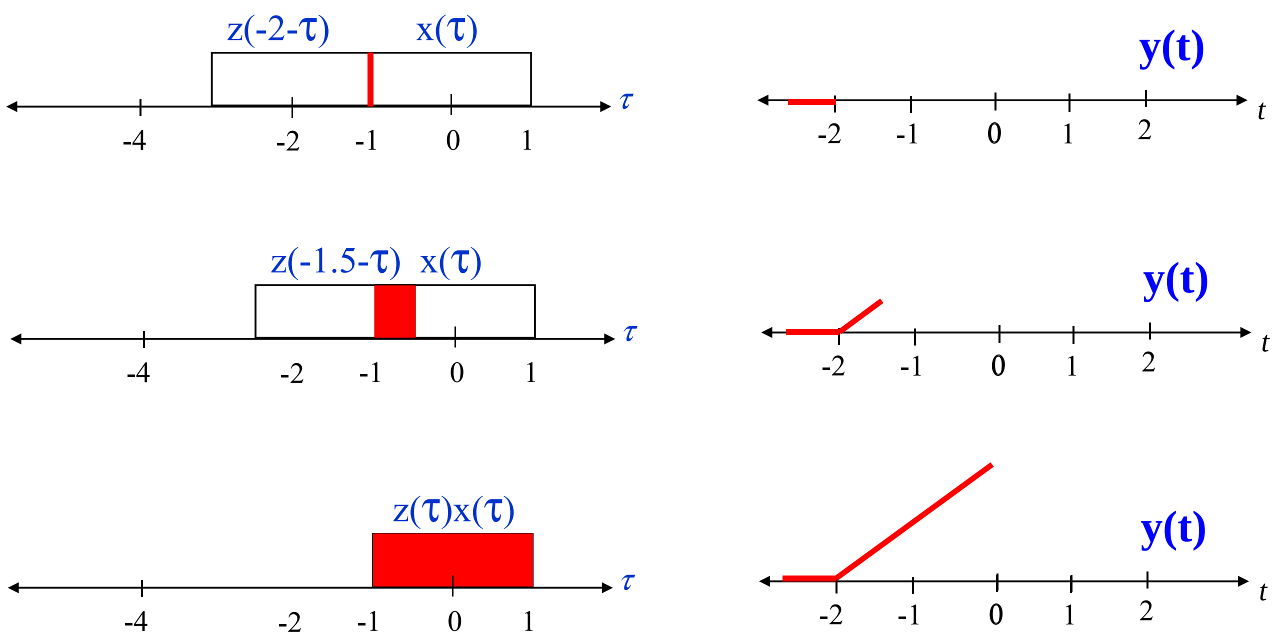
Review of Convolution

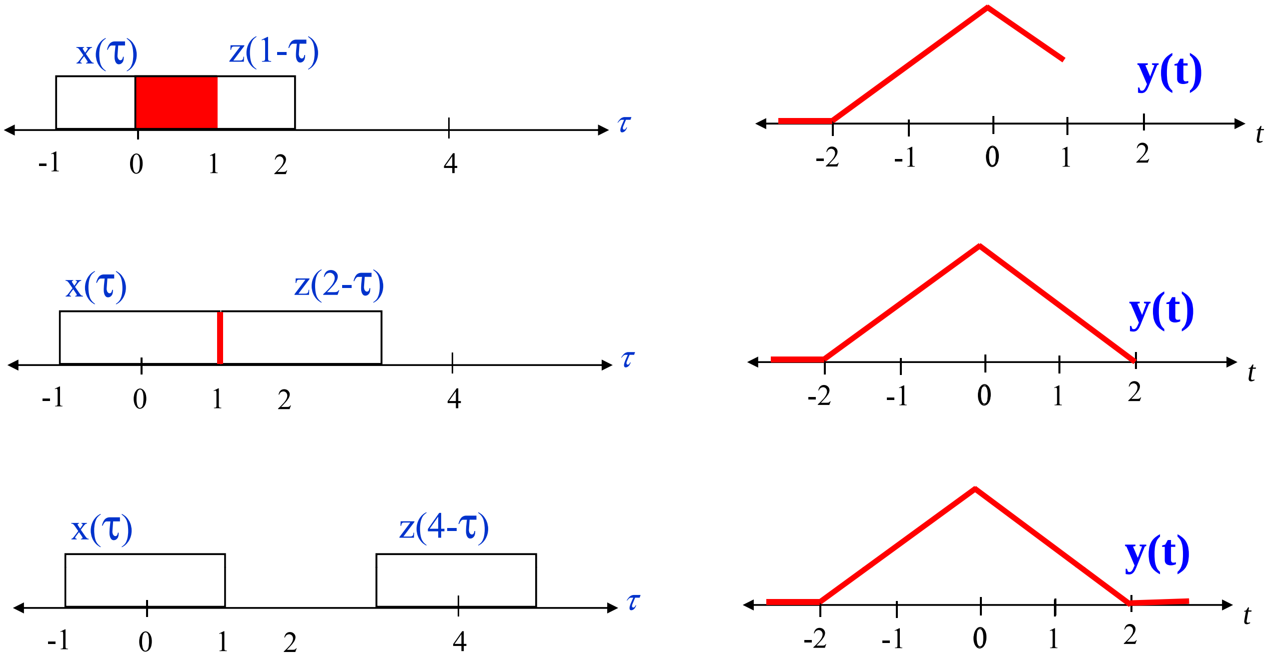
- $y(t) = x(t)*z(t) = \int x(\tau)z(t-\tau)d\tau$
  - Flip one signal and drag it across the other
  - Area under product at drag offset t is $y(t)$.





Review of Convolution (Cont.)





Properties of Convolution

- ❑ Commutative

$$x * y = y * x$$

- ❑ Associative

$$x * (y + z) = x * y + x * z$$

- ❑ Distributive

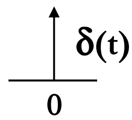
$$(x * y) * z = x * (y * z)$$

Exercise – Convolution

Prove that $x_1(t) * x_2(t) \Leftrightarrow X_1(\omega)X_2(\omega)$

Special Functions

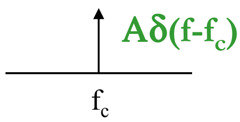
- Dirac delta function



$\Leftrightarrow$ 1

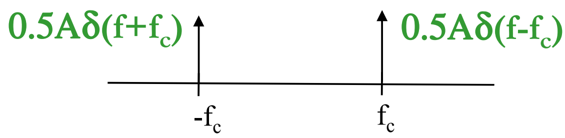
- Exponentials

$Ae^{j2\pi f_c t}$ $\Leftrightarrow$

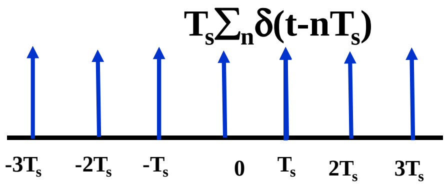


- Sinusoids

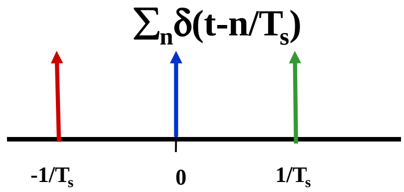
$A\cos(2\pi f_c t)$ $\Leftrightarrow$



- Delta Function Train

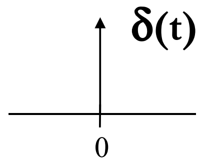


$\Leftrightarrow$

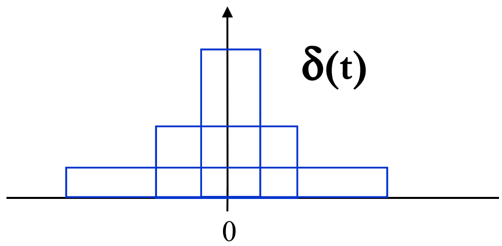


Dirac Delta Function

- Dirac delta function is a mathematical construct that is useful in analyzing signals and filters
- Defined by two equations
  - $\delta(t)$ = infinity, $t = 0$
  - $\int \delta(t)dt = 1$



- Alternatively defined as a limit
  - $\delta(t) = \lim_{\tau \to 0} (1/\tau)\mathrm{rect}(t/\tau)$



Delta Function Properties

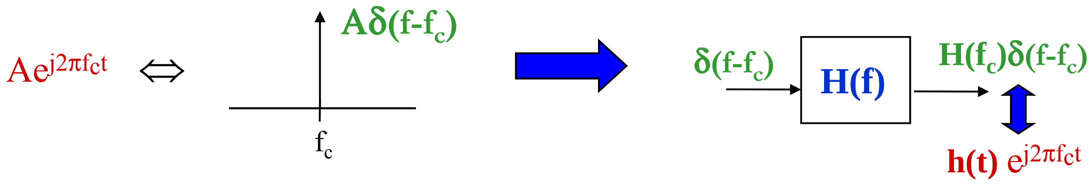
- $v(t) * \delta(t) = v(t)$
- $\delta(t) \Leftrightarrow 1$
- DC signals are δ functions in frequency.
- Integration

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

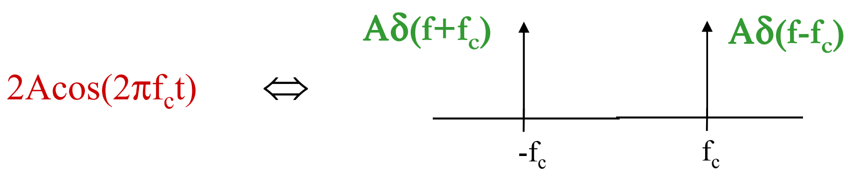
$$\int_{t_1}^{t_2} \delta(t)v(t)dt = \begin{cases} v(0) & t_1 < 0 < t_2 \\ 0 & \text{else} \end{cases}$$

Sinusoids and Exponentials

- Exponentials become a shifted delta

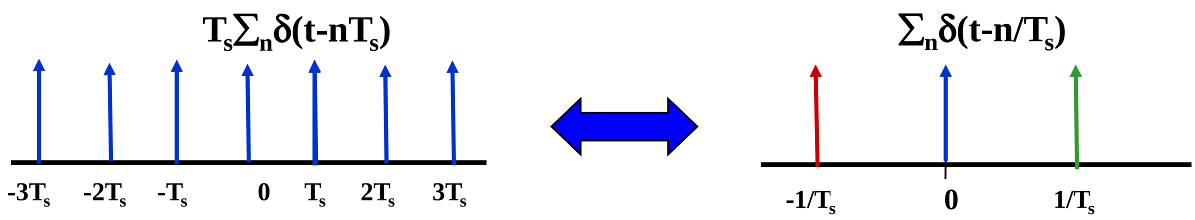


- Sinusoids become two shifted deltas



- Exponentials and sinusoids in time are simple combinations of delta functions in frequency

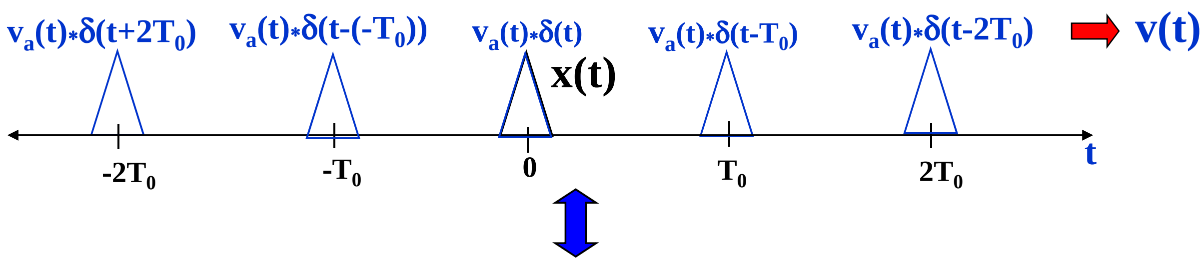
Delta Function Trains (Sampling Function)



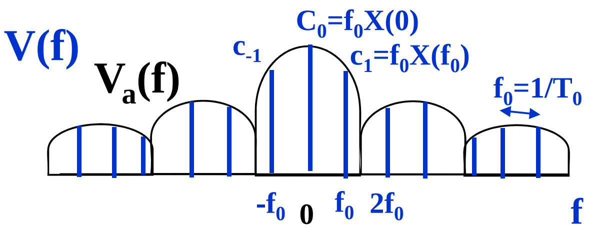
- Delta function trains in time is a delta function train in frequency

Fourier Transforms for Periodic Signals

$$v(t) = v_a(t) * \Sigma_n \delta(t-nT_0) = \Sigma_n v_a(t) * \delta(t-nT_0) = \Sigma_n c_n e^{j2\pi n/T_0}$$



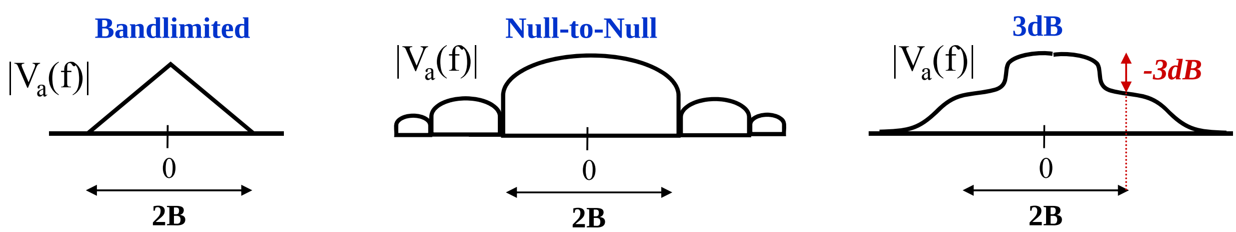
$$V(f) = V_a(f) \bullet (1/T_0)\Sigma_n \delta(f-n/T_0) = \Sigma_n (1/T_0) V_a(n/T_0)\delta(f-n/T_0)$$
$$= \Sigma_n c_n \delta(f-n/T_0)$$



$$c_n = \frac{1}{T_0} V_a\left(\frac{n}{T_0}\right) = f_0 V_a(nf_0)$$

Signal Bandwidth

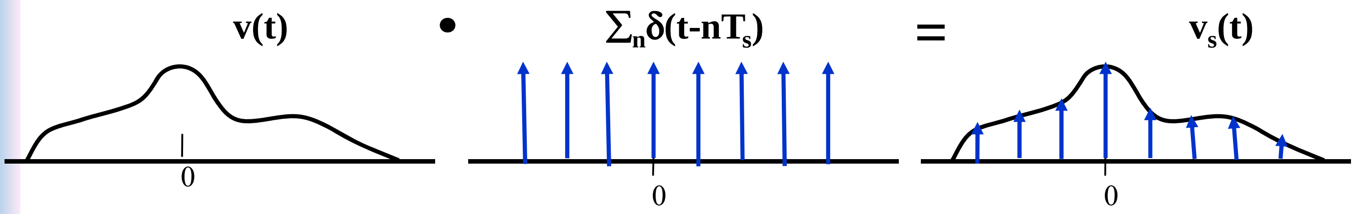
- For bandlimited signals, bandwidth B defined as range of positive frequencies for which $|V_a(f)| > 0$.
- In practice, all signals time-limited
  - Not bandlimited
  - Need alternate bandwidth definition



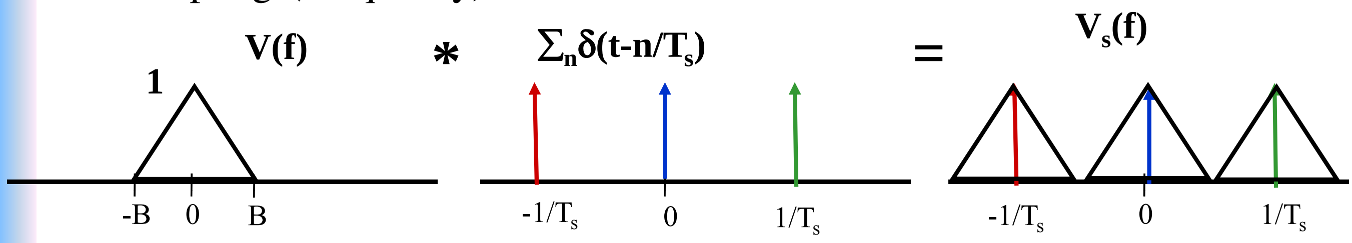
⇒ Signal bandwidth definition depends on its use

Sampling

- Sampling (Time):



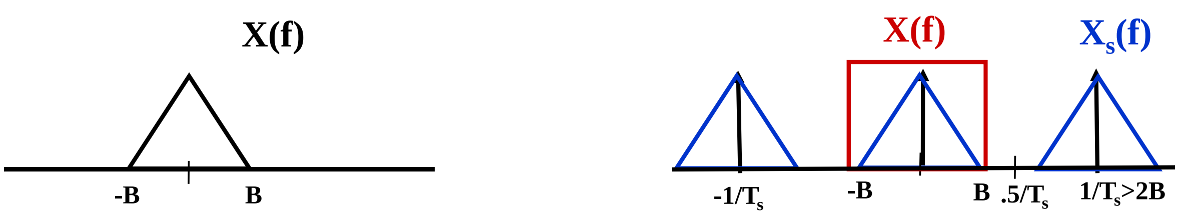
- Sampling (Frequency)



Nyquist: Must sample at $T_s<1/(2B)$ to recreate signal from samples

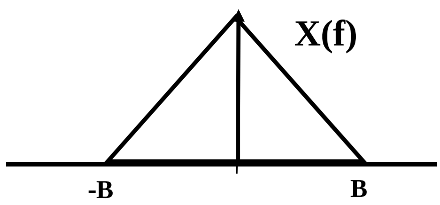
Nyquist Sampling Theorem

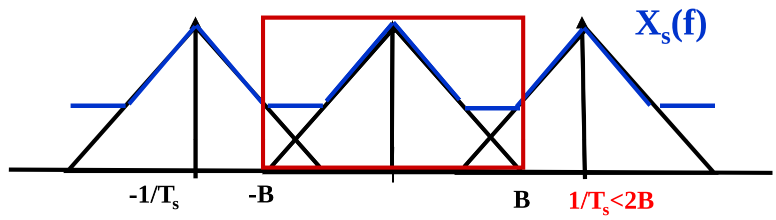
- A bandlimited signal [-B, B] is completely described by samples every $T_s < B/2$ secs.
  - Nyquist rate is 2B samples/sec
- Recreate signal from its samples by using a low pass filter in the frequency domain



Aliasing

- Aliasing occurs when a signal is sampled below its Nyquist rate
  - Repetitions in frequency domain overlap
  - Distortion (aliasing) in frequency domain



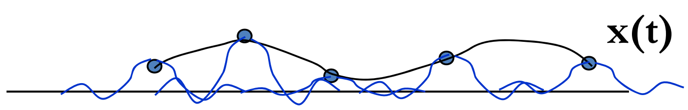


Signal Recovery and Interpolation

- Recover signal in frequency domain by passing sampled signal through a lowpass filter LPF (rect)
- In time domain this becomes convolution of samples with sinc function

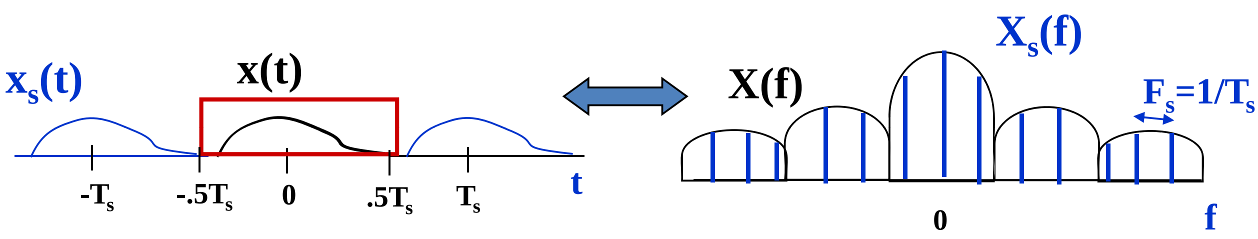
$$x(t) = \sum_n x(nT_s)\,sinc\left(\frac{t - nT_s}{T_s}\right) \Leftrightarrow X_s(f)\,T_s\,rect(fT_s)$$

- Sinc function tracks signal changes between samples



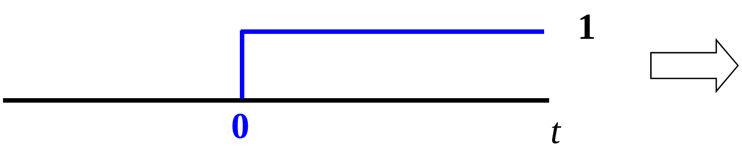
Sampling in Frequency

- By duality, can recover time limited signal by sampling sufficiently fast in frequency
- Sampling in frequency is periodic repetition in time
- Recover time limited signal by windowing



Unit Step Function

- Unit step function u(t):



$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

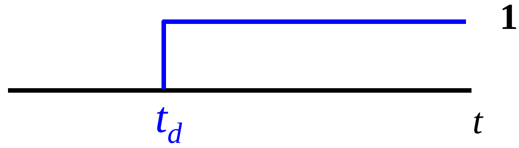
- Fourier transform: u(t) ⇔ 0.5δ(f) + 1/(j2πf)
- Integration

$$\int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t) \Leftrightarrow X(f)U(f) = \frac{1}{2}X(0)\delta(f) + \frac{X(f)}{j2\pi f}$$

- Relation with the unit impulse δ(t)

$$\int_{-\infty}^{t} \delta(\lambda - t_d) d\lambda = \begin{cases} 1 & t > t_d \\ 0 & t < t_d \end{cases}$$

$$= u(t - t_d)$$



Differentiation:

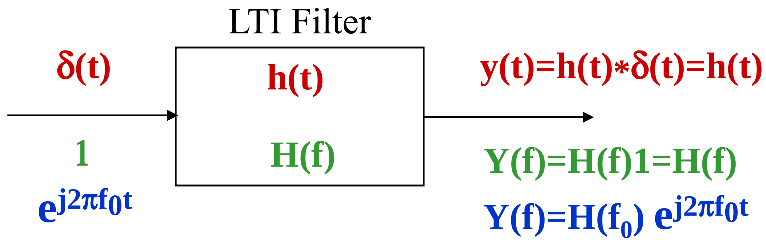
$$\delta(\lambda - t_d) = \frac{d}{dt} u(\lambda - t_d)$$

Summary

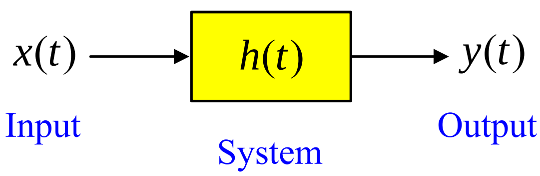
- δ functions, sinusoids and exponentials are key in Fourier analysis
- Delta function train in time is a delta function train in freq.
- Must sample at twice signal BW to recreate signal from samples
- Periodic signals have discrete Fourier transforms consisting of delta functions (frequency sampling)
- Sampling in time is multiplication by delta train: transforms to convolution with delta train

Filter Response

- **Impulse Response** (Time Domain)
  - Filter output in response to a delta input
- **Frequency Response** (Frequency Domain)
  - Fourier transform of impulse response
  - The response of a filter to an exponential input the same exponential weighted by $H(f_0)$



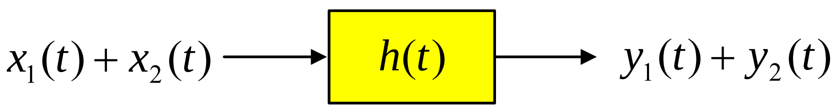
Linear Time-Invariant (LTI) System



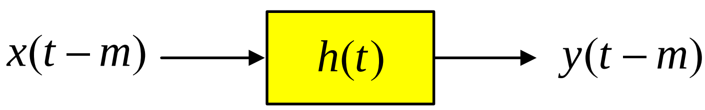
$x(t) \rightarrow y(t)$

$h(t)$ = system response

- **Linear**

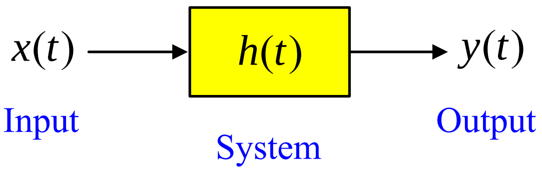


- **Time-invariant**



where m is the amount of time shift.

LTI System



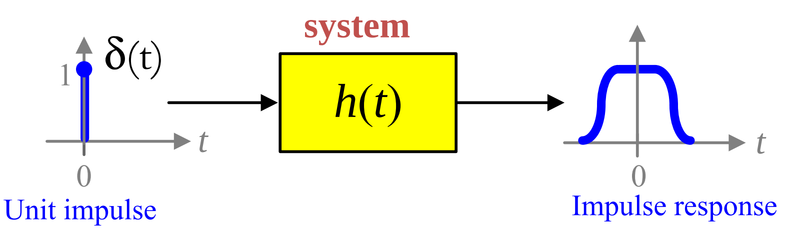
Key:

- Completely characterized by its impulse response, i.e., $h(t)$.
- The output of the system can be expressed in terms of the input and the impulse response as a **convolution**, i.e.,

$$y(t) = x(t) * h(t) = \begin{cases} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \end{cases}$$

$\Rightarrow$ Not hold for a nonlinear system.

Impulse Response

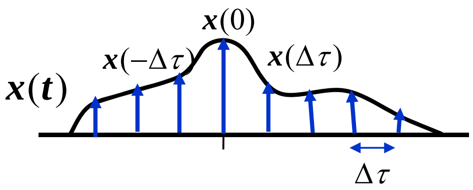


Delta function (unit impulse):

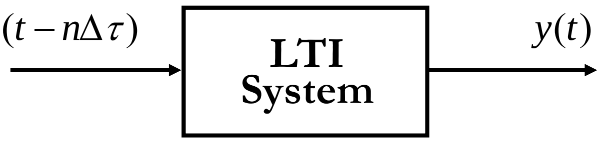
$$\delta[t] = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}$$

- Impulse response $\Rightarrow$ a system response to a delta function
- A system that has a finite number of nonzero outputs in response to a delta function is referred to as a finite impulse response (FIR) system.
- A system that is not FIR is infinite impulse response (IIR).

Filtering as Convolution



$x(t) \approx \sum_{n=-\infty}^{\infty} x(n\Delta\tau)\delta(t-n\Delta\tau)$



$y(t) \approx \sum_{n=-\infty}^{\infty} x(n\Delta\tau)h(t-n\Delta\tau)$

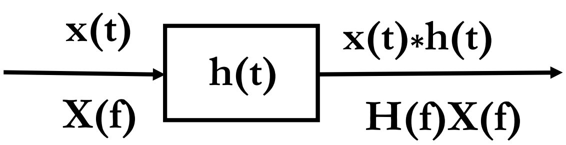
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

- Indicates that the system has memory

Filtering

❑ Filter response to δ(t) is impulse response

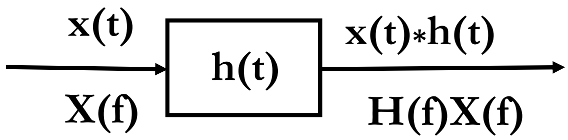
❑ For any input x(t), filter output is x(t)*h(t)



⇒ Much easier to study filtering in the frequency domain

Frequency Response

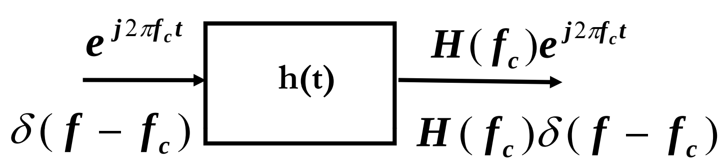
- Fourier transform of impulse response
  - Typically complex: amplitude and phase response



$$H(f) = |H(f)|\, e^{j\angle H(f)}$$

Measured using eigenfunctions

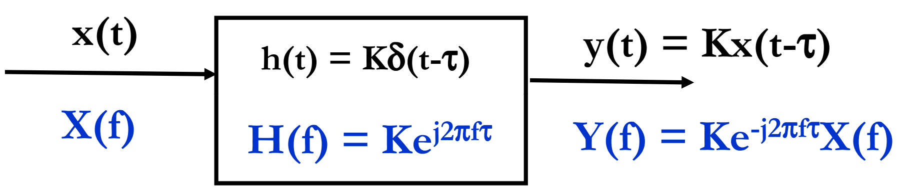
- Exponential eigenfunctions



$$\int_{-\infty}^{\infty} h(\tau)e^{j2\pi f_c(t-\tau)}d\tau = e^{j2\pi f_c t}\underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f_c \tau}d\tau}_{H(f_c)}$$

Distortion

- Distortionless transmission
  - The output signal equals the input except for amplitude scaling and/or delay ⇒ same "shape"



- A system giving distortion less must have constant amplitude response and negative linear phase shift, i.e.,

$$|H(f)| = |K| \quad \text{and} \quad \angle H(f) = -2\pi t_d f \pm m180^\circ$$

Must pass through the origin

- ❑ Communication systems always produce some amount of signal distortion
- ❑ Three major types of distortion:
  - Amplitude distortion ⇒ occur when

$$|H(f)| \neq |K|$$

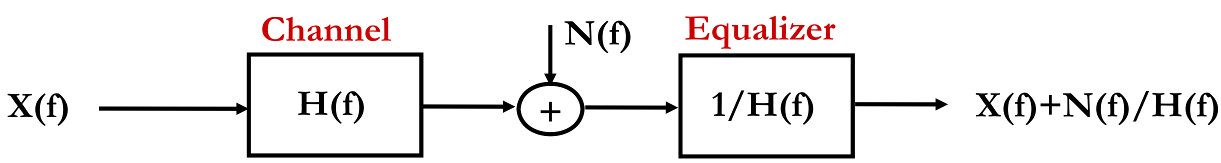
  - Phase distortion ⇒ occur when

$$\angle H(f) \neq -2\pi t_d f \pm m180^\circ$$

} Linear distortion

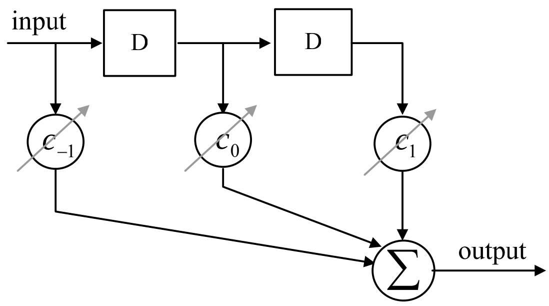
  - Nonlinear distortion ⇒ when systems include nonlinear elements

- ❑ Linear distortion can be cured by the use of equalizers



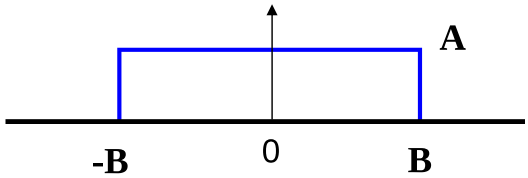
  - May enhance noise power (e.g., if H(f) → 0 for some f's)

- ❑ Equalizer:
  - (fixed) tapped-delay-line equalizer or transversal filter
  - Adaptive equalizer ⇒ compensate for changing channel characteristics

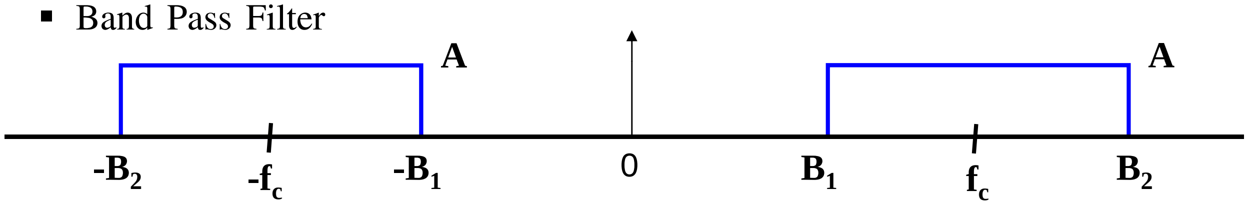


Ideal Filters

- ❑ Used to separate an information signal from unwanted signals.
- ❑ Has the characteristics of distortionless transmission
- ❑ Difficult to realize
- ❑ Examples:
  - Low Pass Filter



  - Band Pass Filter



Summary

- ❑ Communication channels and filters are LTI systems
- ❑ The output of an LTI system is the convolution of its impulse response with the input signal
- ❑ LTI system output in frequency domain is the product of the input Fourier transform with the system frequency response
- ❑ An LTI system is distortionless if it only yields amplitude change and/or a delay (linear phase shift)
- ❑ Most communication channels introduce distortion
- ❑ Equalizers compensate for channel distortion but might enhance receiver noise
- ❑ Most communication systems employ one or more filters

สหสัมพันธ์

- สหสัมพันธ์ (correlation) เป็นเครื่องมือทางคณิตศาสตร์ที่ใช้ในการหาความสัมพันธ์ระหว่างสัญญาณสองสัญญาณว่ามีความสอดคล้องกันมากน้อยเพียงใด
  - ถ้าสัญญาณมีความสัมพันธ์กันมาก ผลลัพธ์ที่ได้ก็จะมีค่ามาก (และในทางตรงกันข้าม)
- สหสัมพันธ์แบ่งออกเป็น 2 ประเภท
  - สหสัมพันธ์ข้าม (cross-correlation)
  - อัตสหสัมพันธ์ (auto-correlation)

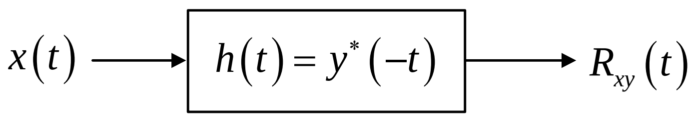
สหสัมพันธ์ข้าม

- ถ้าให้ x(t) และ y(t) เป็นสัญญาณพลังงาน ⇒ ฟังก์ชันสหสัมพันธ์ข้าม $R_{xy}(\tau)$ ณ เวลาล่า (lag time) τ หาได้จาก

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t+\tau)\, y^*(t)\, dt = \int_{-\infty}^{\infty} x(t)\, y^*(t-\tau)\, dt$$

  - คล้ายคอนโวลูชัน

$$R_{xy}(t) = x(t) * y^*(-t) = \int_{-\infty}^{\infty} x(\tau)\, y^*(-(t-\tau))\, dt == \int_{-\infty}^{\infty} x(\tau)\, y^*(\tau - t)\, dt$$



อัตสหสัมพันธ์

❑ ฟังก์ชันอัตสหสัมพันธ์ของ x(t) เขียนแทนด้วย $R_{xx}(\tau)$ นิยามโดย

$$R_{xx}(\tau)=\int_{-\infty}^{\infty}x(t+\tau)x^*(t)\,dt=\int_{-\infty}^{\infty}x(t)x^*(t-\tau)\,dt=x(t)*x^*(-t)$$

คุณสมบัติที่สำคัญ

- มีความสมมาตร $\Rightarrow$ ถ้า $x(t)$ เป็นฟังก์ชันค่าจริง $\Rightarrow$ $R_{xx}(\tau)=R_{xx}(-\tau)$ เป็นฟังก์ชันคู่
  แต่ถ้า $x(t)$ เป็นฟังก์ชันเชิงซ้อน $\Rightarrow$ $R_{xx}^*(\tau)=R_{xx}(-\tau)$ เป็นฟังก์ชันเฮอร์มิเชียน (hermitian)
- ค่าของฟังก์ชันอัตสหสัมพันธ์ที่เวลา $\tau=0$ จะมีค่าเท่ากับพลังงานของสัญญาณ

$$E_x=R_{xx}(0)=\int_{-\infty}^{\infty}|x(t)|^2\,dt \qquad \text{เนื่องจาก } |x(t)|^2=x(t)x^*(t)$$

- ค่าของฟังก์ชันอัตสหสัมพันธ์ $R_{xx}(0)\ge|R_{xx}(\tau)|$

- อัตสหสัมพันธ์ของผลรวมของฟังก์ชันที่ไม่สัมพันธ์กัน (uncorrelated function) มีค่าเท่ากับผลรวมของฟังก์ชันอัตสหสัมพันธ์ของแต่ละฟังก์ชัน
- ถ้ากำหนดให้ $n(t)$ เป็นสัญญาณรบกวนสีขาว ฟังก์ชันอัตสหสัมพันธ์ของ $n(t)$ มีค่าเท่ากับ

$$R_{nn}(\tau)=\begin{cases}K\delta(\tau), & \tau=0\\ 0, & \text{else}\end{cases} \qquad \text{เมื่อ } K \text{ เป็นค่าคงตัว}$$

- ฟังก์ชันอัตสหสัมพันธ์ $R(\tau)$ มีความสัมพันธ์กับความหนาแน่นสเปกตรัมกำลัง $G(f)$ ผ่านทางการแปลงฟูเรียร์ตามทฤษฎีบทของวินเนอร์คินชิน (Wiener–Khinchin theorem) ดังนี้

$$G(f)=\int_{-\infty}^{\infty}R(\tau)e^{-j2\pi f\tau}\,d\tau \;\Leftrightarrow\; R(\tau)=\int_{-\infty}^{\infty}G(f)e^{j2\pi f\tau}\,df \quad \text{(คู่การแปลงฟูเรียร์)}$$