Communication Engineering Systems

Signal and Spectra (2)

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"Actions speak louder than words"

Outline

- ❑ Introduction
- ❑ Phasor
- ❑ Periodic and Aperiodic Signals
- ❑ Fourier Series
- ❑ Fourier Transform

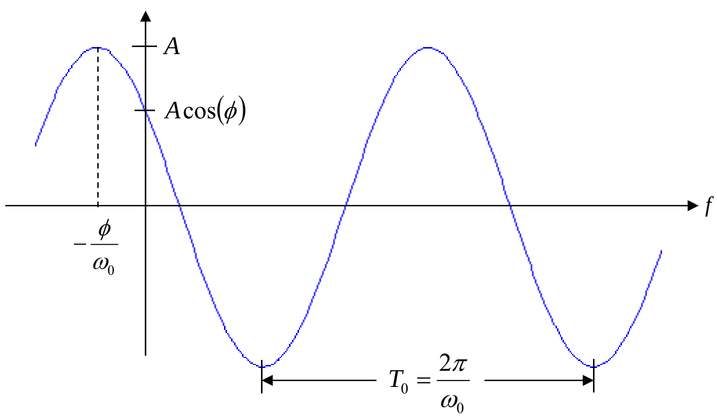
Introduction

- The signal in time domain can be represented in the frequency domain, where it is viewed as consisting of sinusoidal components at various frequencies.
- This frequency-domain description is called the **spectrum**.
- Line spectra are based on the Fourier series (FS) expansion of periodic continuous-time signals.
- Continuous spectra are based on the Fourier transform (FT) of aperiodic continuous-time signals.

Sinusoidal Waveform

Consider a sinusoidal or AC waveform:

$$v(t) = A\cos(\omega_0 t + \phi)$$



where

A = peak value or amplitude

$\omega_0 = 2\pi f_0$ = radian frequency

ϕ = phase angle

Fundamental frequency $\Rightarrow f_0 = \dfrac{1}{T_0}$

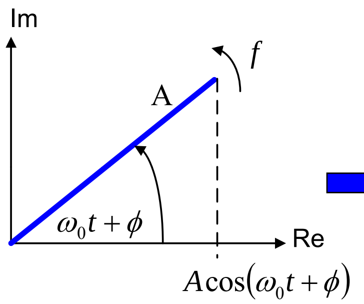
Phasor

- v(t) $\Rightarrow$ an eternal sinusoidal $\Rightarrow$ usually will be represented by a complex exponential or **phasor**.
- Euler's theorem:

$$e^{\pm j\theta} = \cos(\theta) + j\sin(\theta) \qquad j = \sqrt{-1}$$

$$A\cos(\omega_0 t + \phi) = A\,\mathrm{Re}\left[e^{j(\omega_0 t + \phi)}\right] = A\,\mathrm{Re}\Big[\underbrace{e^{j\omega_0 t} e^{j\phi}}\Big]$$

can be viewed as a rotating vector in a complex plane

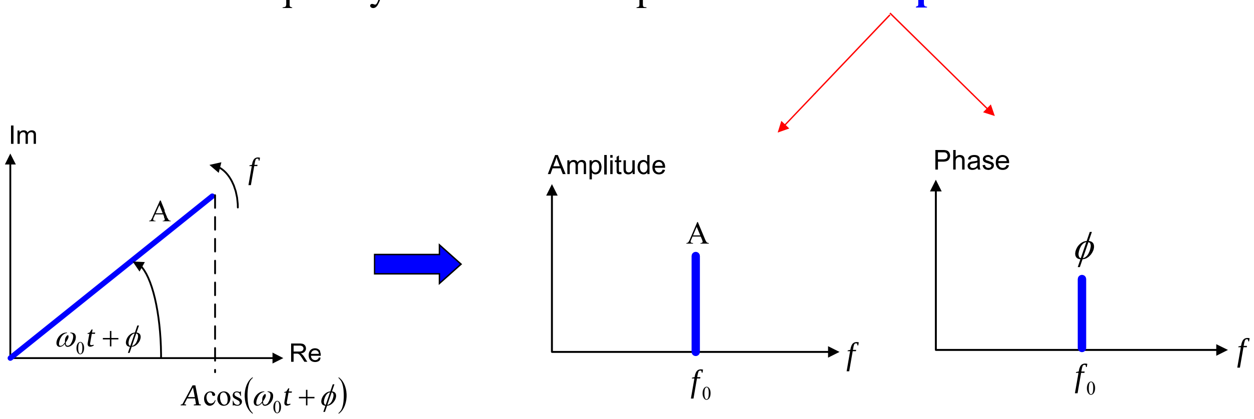


Three parameters completely specify the phasor

A, ω_0, ϕ

Phasor (Cont.)

- To describe the same phasor in frequency domain, we must associate the corresponding amplitude and phase with the particular frequency f_0.
- A suitable frequency domain description $\Rightarrow$ **line spectrum**



Phasor (Cont.)

- Four conventions for constructing the line spectrum:
  - The phase angle is normally measured wrt. cosine wave

$$\sin(\omega t) = -\cos(\omega t + 90°)$$

  - The cyclical frequency f is used for the x-axis
  - The amplitude is always positive

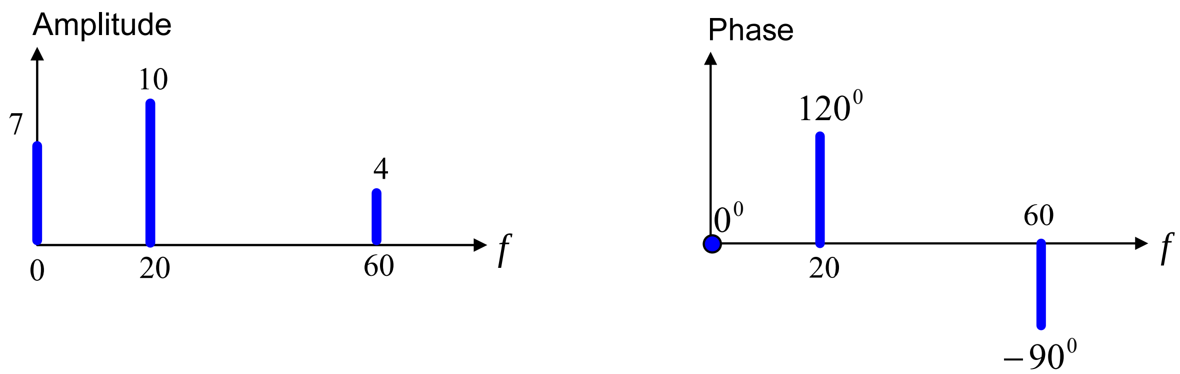
$$-A\cos(\omega t) = A\cos(\omega t \pm 180°)$$

  - The phase angle is usually measured in degree
- Practically, the amplitude spectrum conveys more information than the phase spectrum.

Phasor (Cont.)

Ex: Plot the line spectra of $w(t) = 7 - 10\cos(40\pi t - 60°) + 4\sin(120\pi t)$

$$\begin{aligned} w(t) &= 7 - 10\cos(40\pi t - 60°) + 4\sin(120\pi t) \\ &= 7\cos(2\pi 0t) + 10\cos(2\pi 20t - 60° + 180°) - 4\cos(2\pi 60t + 90°) \\ &= 7\cos(2\pi 0t) + 10\cos(2\pi 20t - 60° + 180°) + 4\cos(2\pi 60t + 90° - 180°) \end{aligned}$$



One-side or positive frequency line spectra

Phasor (Cont.)

- **Two-sided** line spectra (more valuable)

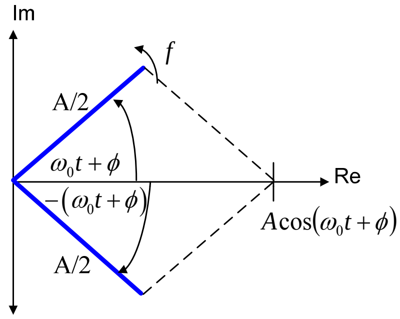
$$\mathrm{Re}[z] = \frac{1}{2}\left[z + z^*\right]$$ $\Rightarrow$ z is a complex quantity with complex conjugate z*

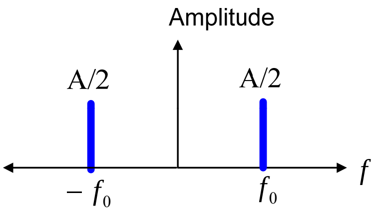
$$z = Ae^{j\phi}e^{j\omega t} \quad \Rightarrow \quad z^* = Ae^{-j\phi}e^{-j\omega t}$$

a pair of conjugate phasor

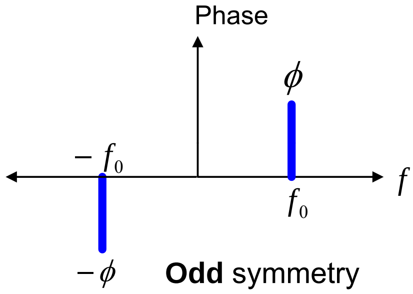
Thus:

$$A\cos(\omega_0 t + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega t} + \frac{A}{2}e^{-j\phi}e^{-j\omega t}$$





Even symmetry



Odd symmetry

Type of Signals

- Type I
  - Continuous-time signal (analog signal)
  - Discrete-time signal (digital signal)
- Type II
  - Periodic signal
  - Aperiodic signal
- Type III
  - Deterministic signal
  - Random signal

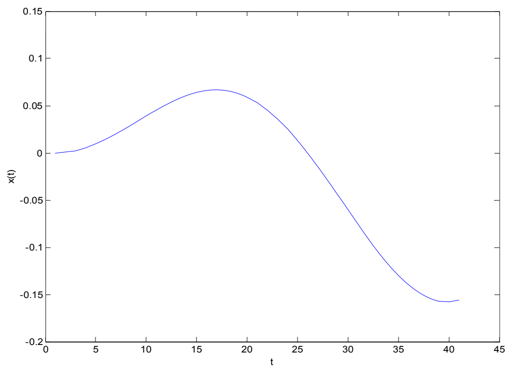
Continuous/Discrete-Time Signal

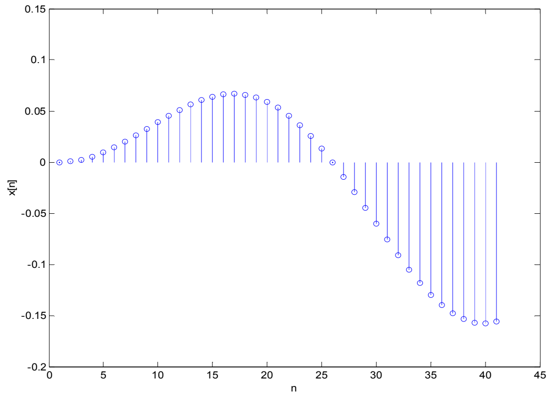
A *continous-time* signal $x(t)$ occurs at all time

A *discrete-time* signal $x[n]$occurs only at instants of time $t = nT$, i.e.,

$$x[n] = x(nT),$$

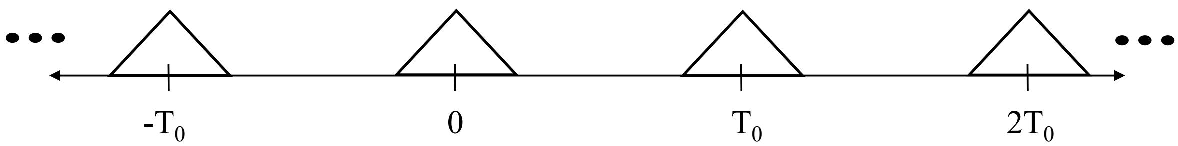
where T = sampling period and n can be positive or negative integers





Periodic & Aperiodic Signal

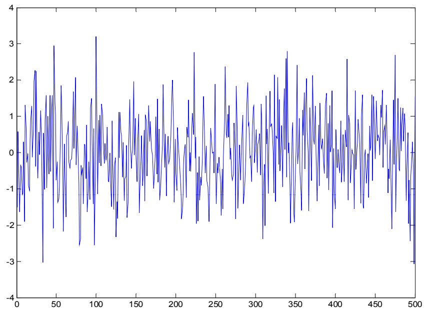
- $x_p(t)$ **periodic** if exists T such that $x_p(t)=x_p(t+T)$ for all t.



  - Fully specified its behavior over any one period
- Smallest such T is fundamental period T_0
  - Any integer multiple of T_0 is a period of $x_p(t)$
  - Fundamental period defined as $f_0 = 1/Tz$
- **Aperiodic** signals are not periodic

Deterministic/Random Signal

- **Deterministic** signal x(t), x[n] can be specified at all time
- **Random** signal x(t), x[n] has uncertainty of its occurrence. It typically belongs to an ensemble of signals with certain probability of occurrence. An ensemble of such signal is called random process



Gaussian noise
with mean = 0,
variance = 1

Periodic Signals – Average Power

- Average value:

$$\langle |v(t)| \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} v(t)dt \quad \xrightarrow{\text{Periodic}} \quad \langle v(t) \rangle = \frac{1}{T_0} \int_{T_0} v(t)dt$$

- Average power $\Rightarrow$ Real and nonnegative

$$P = \left\langle |v(t)|^2 \right\rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |v(t)|^2 dt \quad \xrightarrow{\text{Periodic}} \quad P = \frac{1}{T_0} \int_{T_0} |v(t)|^2 dt$$

Periodic Signals – Average Power (Cont.)

Ex: Find the average value and power of $v(t) = A\cos(\omega t + \phi)$

Average value: $\langle v(t) \rangle = \frac{1}{T_0}\int_0^{T_0} A\cos(\omega t + \phi)dt = \frac{A}{T_0}\underbrace{\int_0^{T_0}\cos(\omega t + \phi)dt}_{=0}$

Average power:

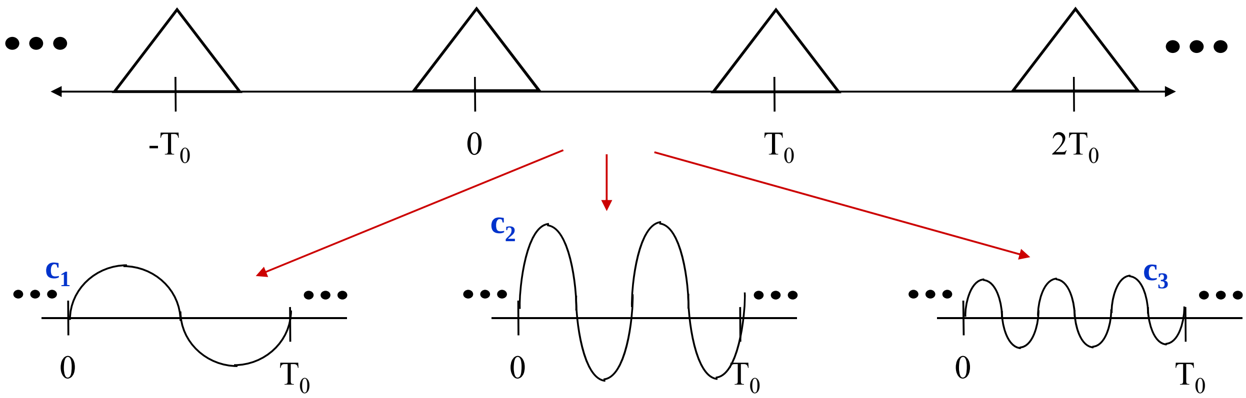
$$P = \langle |v(t)|^2 \rangle = \frac{1}{T_0}\int_0^{T_0} A^2\cos^2(\omega t + \phi)dt$$
$$= \frac{A^2}{2T_0}\int_0^{T_0}\{1 + \cos(2\omega t + 2\phi)\}dt$$
$$= \frac{A^2}{2T_0}T_0 + \frac{A^2}{2T_0}\underbrace{\int_0^{T_0}\cos(2\omega t + 2\phi)dt}_{=0} = \frac{A^2}{2}$$

Transform Representation

- Electrical communication signals are time-varying quantities, e.g., current and voltage.
- These signals physically exist in time domain.
- Frequency domain representation is very useful in communication systems because it allows simple computation in many cases.
- Two types of transforms:
  - Fourier series (FS) $\Rightarrow$ Periodic signal
  - Fourier transform (FT) $\Rightarrow$ Aperiodic signal

Fourier Series (FS)

- Decompose periodic signals into sum of sinusoidal waveforms, or equivalently, rotating phasors.



Fourier Series (Cont.)

- Let $v(t)$ be a *power* (periodic) signal with period of $T_0 = 1/f_0$, its exponential FS expansion is

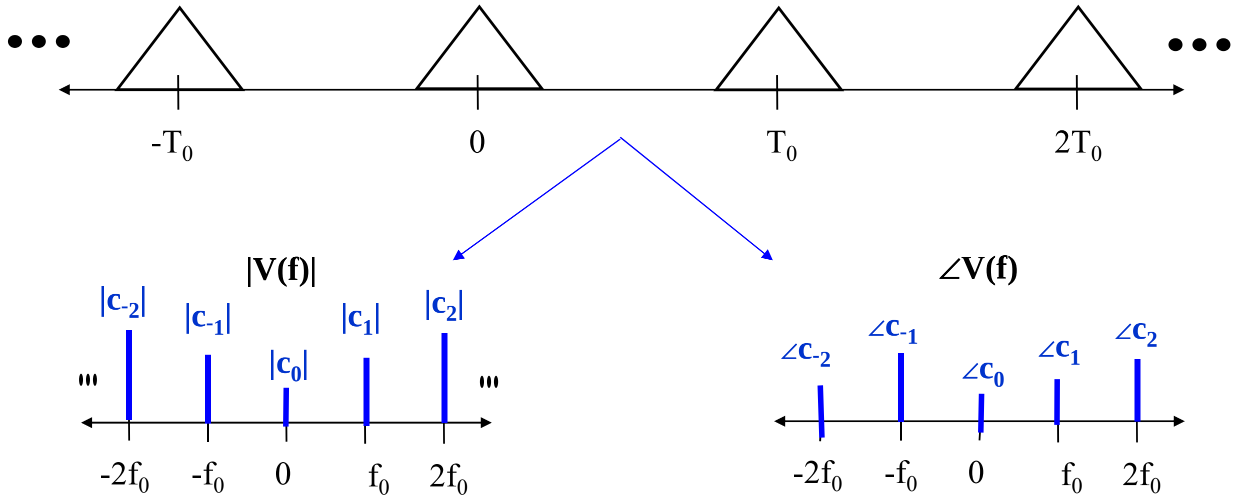
$$v(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t} \qquad n = 0, 1, 2, 3, \ldots$$

where

$$c_n = \frac{1}{T_0}\int_{T_0} v(t) e^{-j2\pi n f_0 t} dt \quad \Rightarrow \quad c_n = |c_n| e^{j\angle c_n}$$

- $\{c_n\}$ are the FS coefficients $\Rightarrow$ represent the *frequency components* of the periodic signal.

Fourier Series (Cont.)



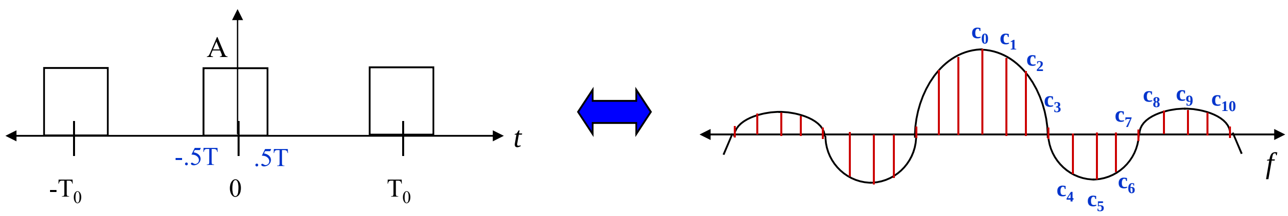
| c_n | represents the amplitude spectrum as a function of *f*

Lc_n represents the phase spectrum as a function of *f*

Example: Periodic Square Wave

Find c_n

Symmetric coefficients ($c_n = c^*_{-n}$)
Infinite frequency content



$$c_n = Af_0 T sinc(nf_0 T) \Rightarrow sinc(x) = \frac{\sin \pi x}{\pi x} = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x = 1, 2, 3, \ldots \end{cases}$$

Example $\Rightarrow$ $\frac{1}{T}\int_{-T/2}^{T/2} e^{j2\pi ft}dt = \frac{1}{j2\pi fT}\left(e^{j\pi fT} - e^{-j\pi fT}\right) = \frac{\sin(\pi fT)}{\pi fT} = sinc(fT)$

Properties of FS

- The spectral lines have uniform spacing f_0
- The DC component equals the average value of the signal

$$c(0) = \frac{1}{T_0} \int_{T_0} v(t)\,dt = \langle v(t) \rangle$$

- If $v(t)$ is a real (noncomplex) function of time, then

$$c_{-n} = c_n^* = |c_n| e^{-jLc_n}$$

$| c_{-n} | = | c_n |$ ⇒ Amplitude spectrum ⇒ **Even** symmetry

$Lc_{-n} = -Lc_n$ ⇒ Phase spectrum ⇒ **Odd** symmetry

FS – Real Signal

- A real signal v(t) can also be expressed as

$$v(t) = c_0 + 2\sum_{n=1}^{\infty} |c_n| \cos(2\pi n f_0 t + \angle c_n)$$

⇒ Trigonometric FS (One-sided spectrum)

- Alternate FS representation

$$v(t) = a_0 + 2\sum_{n=1}^{\infty} \left[a_n \cos(2\pi n f_0 t) + b_n \sin(2\pi n f_0 t) \right]$$

$$a_n = \frac{1}{T_0} \int_{T_0} v(t) \cos(2\pi n f_0 t)\,dt \qquad b_n = \frac{1}{T_0} \int_{T_0} v(t) \sin(2\pi n f_0 t)\,dt$$

Key Properties of FS

- Linearity
- Multiplication $\Rightarrow$ Multiplication in time leads to convolution of FS
- Time Shifting $\Rightarrow$ Time shift leads to linear phase shift in FS
- Time Reversal $\Rightarrow$ Time reversal leads to index reversal
- Time Scaling $\Rightarrow$ Time scaling leads to frequency stretching
- Conjugation: $x_p^*(t) \Leftrightarrow \{c_{-n}^*\}$
- Parseval's relation:
  - Energy contained in FS

$$\frac{1}{T_0}\int_{T_0} |v(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Prove it !

Aperiodic Signal – Energy

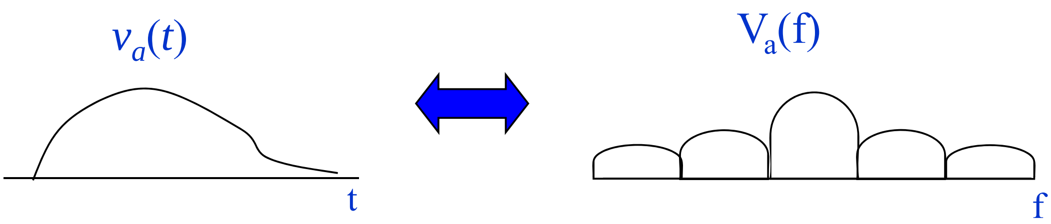
- Aperiodic signal $v_p(t)$ is time-limited signal.
- The average of $|v_p(t)|$ or $|v_p(t)|^2$ over all time equals zero.
- The concept of energy is needed.
- Energy:

$$E = \int_{-\infty}^{\infty} |v(t)|^2 \, dt$$

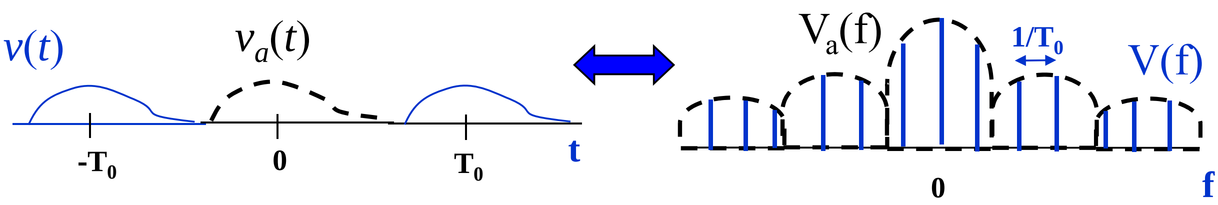
$\Rightarrow$ Total area under the curve of $|v(t)|^2$

Fourier Transform (FT)

- ❑ Used for a continuous-time aperiodic signal $v_a(t)$
- ❑ Represent the spectral components
- ❑ Provide a one-to-one mapping between time and frequency domains.



FS to FT

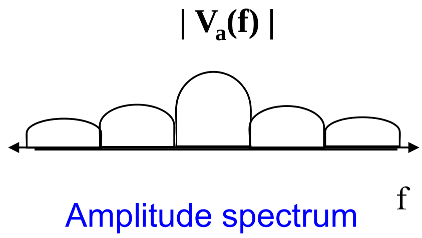


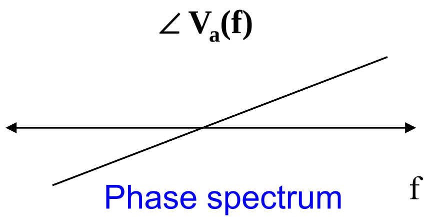
- ❑ Repeat $v_a(t)$ every T_0 seconds to get $v(t)$
- ❑ FS coefficients separated in frequency by $f_0 = 1/T_0$
- ❑ As $T_0 \rightarrow \infty$, samples in frequency domain become a continuous signal in f

Fourier Transform Pair

$$V_a(f) = \int_{-\infty}^{\infty} v_a(t) e^{-j2\pi ft} dt$$

$$v_a(t) = \int_{-\infty}^{\infty} V_a(f) e^{j2\pi ft} df$$

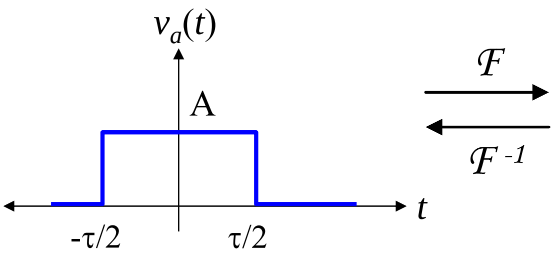




Real signal:

$V_a(-f) = V_a^*(f)$, $|V_a(f)| = |V_a(-f)|$ and $\angle V_a(-f) = -\angle V_a(f)$

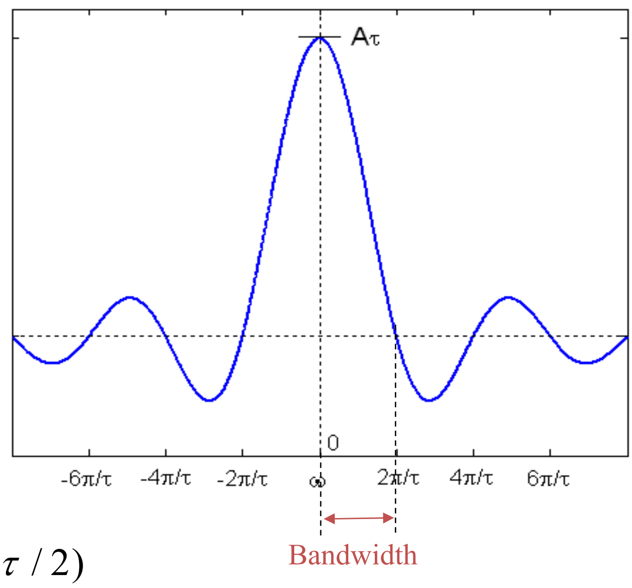
Example – Rectangular Pulse



$$V_a(\omega) = \int_{-\tau/2}^{\tau/2} A e^{-j\omega t} dt$$

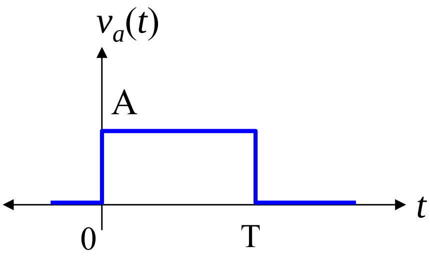
$$= A\left\{\frac{e^{-j\omega t}}{-j\omega}\right\}_{t=-\tau/2}^{\tau/2}$$

$$= \frac{A}{j\omega}\left\{e^{+j\omega\tau/2} - e^{-j\omega\tau/2}\right\} = \frac{2A}{\omega}\sin(\omega\tau/2)$$

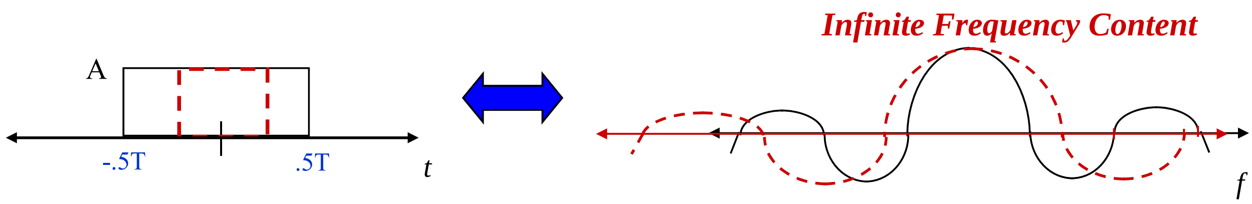


Exercise

Find the FT of a given $v_a(t)$. Hint: $\sin(\theta) = \frac{1}{2j}\left\{e^{j\theta} - e^{-j\theta}\right\}$



Rectangular Pulse



$$v_a(t) = Arect(t/T) \Leftrightarrow V_a(f) = ATsinc(fT)$$

- ❑ Rectangular pulse is a time window
- ❑ Shrinking time axis causes stretching of frequency axis
- ❑ Signals cannot be both time-limited and bandwidth-limited

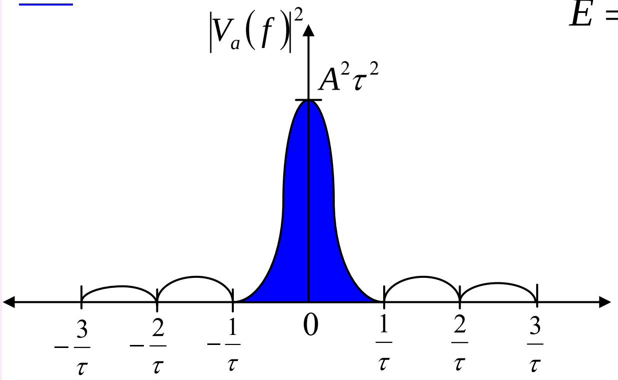
Rayleigh's Energy Theorem

- Analogous to Parseval's power theorem

$$E = \int_{-\infty}^{\infty} V_a(f) V_a^*(f)\,df = \int_{-\infty}^{\infty} |V_a(f)|^2\,df$$

$$V_a(f) = A\tau \cdot sinc(f\tau)$$

Ex:



$$E = \int_{-1/\tau}^{1/\tau} |V_a(f)|^2\,df$$

$$= \int_{-1/\tau}^{1/\tau} (A\tau)^2 sinc^2(f\tau)\,df = 0.92A^2\tau$$

More than 90% of the total energy lie between -1/τ and 1/τ

Properties of FT

- Superposition

$$\sum_k a_k v_{a,k}(t) \Leftrightarrow \sum_k a_k V_{a,k}(f)$$

- Time delay

$$v_a(t - t_d) \Leftrightarrow V_a(f) e^{-j2\pi f t_d}$$

- Scale change

$$v_a(\beta t) \Leftrightarrow \frac{1}{|\beta|} V_a\left(\frac{f}{\beta}\right)$$

Time scaling ⇒ contracting in time yields expansion in frequency

Properties of FT (Cont.)

- Frequency translation and modulation

$$v_a(t)e^{j2\pi f_c t} \Leftrightarrow V_a(f - f_c)$$

Ex:

$$v_a(t)\cos(2\pi f_c t + \phi) \Leftrightarrow \frac{e^{j\phi}}{2}V_a(f - f_c) + \frac{e^{-j\phi}}{2}V_a(f + f_c)$$

- Differentiation and integration

$$\frac{d^n}{dt^n}v_a(t) \Leftrightarrow (j2\pi f)^n V_a(f) \qquad (-jt)v_a(t) \Leftrightarrow \frac{d}{d\omega}V_a(\omega)$$

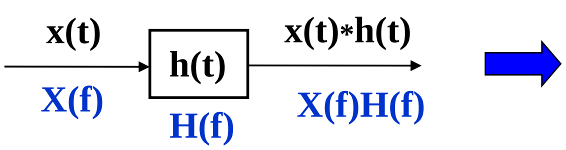
$$\int_{-\infty}^{\infty} v_a(\lambda)d\lambda \Leftrightarrow \frac{1}{j2\pi f}V_a(f)$$

Properties of FT (Cont.)

- DC property

$$V_a(0) = \int_{-\infty}^{\infty} v_a(t)dt, \text{ similarly } v_a(0) = \int_{-\infty}^{\infty} V_a(f)df$$

- Convolution
  - Become multiplication in frequency
  - Define output of linear time-invariant (LTI) filters $\Rightarrow$ easier to analyze with FTs

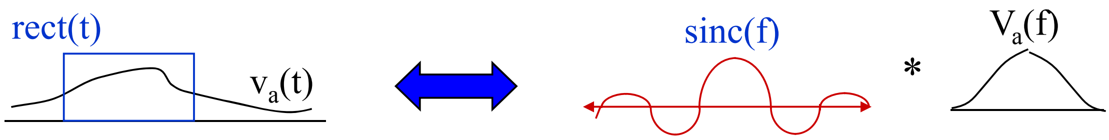


$$x(t) * h(t) \Leftrightarrow X(f)H(f)$$

Properties of FT (Cont.)

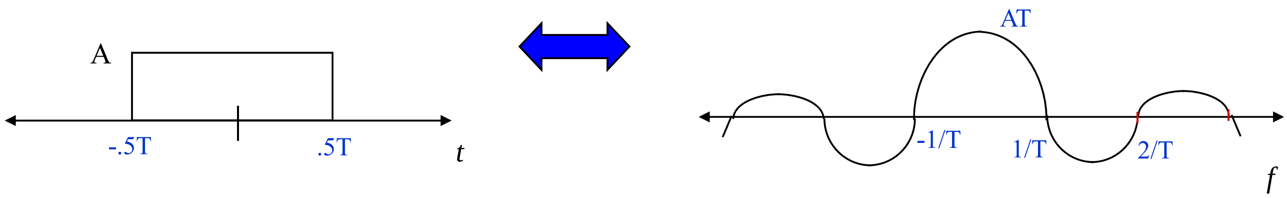
- Multiplication
  - Becomes complicated convolution in frequency
  - Mod/Demod often involves multiplication

$$v(t)w(t) \Leftrightarrow V(f) * W(f)$$

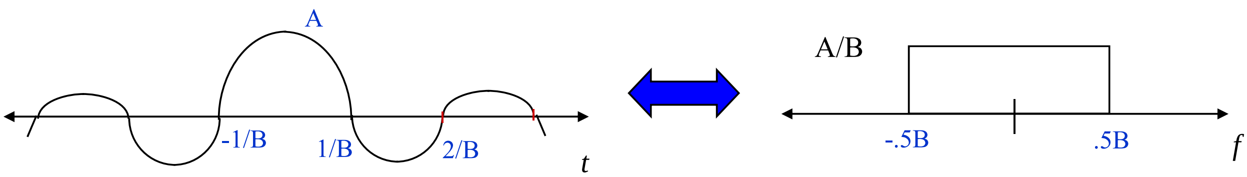


- Duality
  - Operations in time lead to dual operations in frequency
  - Fourier transform pairs are duals of each other

Duality – Rectangular and Sinc Pulses



$$x(t) = A\,rect(t/T) \Leftrightarrow X(f) = A\,T\,sinc(fT)$$



$$x(t) = A\,\mathrm{sinc}(Bt) \Leftrightarrow X(f) = \frac{A}{B}\mathrm{rect}(f/B)$$

Useful FT Pairs

$\delta(t) \Leftrightarrow 1$

$\delta(t-t_0) \Leftrightarrow e^{-j\omega_0 t}$

$1 \Leftrightarrow 2\pi\delta(\omega)$

$e^{j\omega_0 t} \Leftrightarrow 2\pi\delta(\omega-\omega_0)$

$\cos(\omega_0 t) \Leftrightarrow \pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$

$\sin(\omega_0 t) \Leftrightarrow -j\pi\delta(\omega-\omega_0)+j\pi\delta(\omega+\omega_0)$

$u(t) \Leftrightarrow \pi\delta(\omega)+\frac{1}{j\omega}$

$e^{-at}u(t) \Leftrightarrow \frac{1}{j\omega+a}, \quad a>0$

$e^{-a|t|} \Leftrightarrow \frac{2a}{\omega^2+a^2}, \quad a>0$

Additional FT Comments

- Amplitude conveys information about signal's frequency content
  - Phase conveys little insight, except that a time delay results in a linear phase shift
- Dirichlet conditions are sufficient for FT to exist (not necessary)
  - Signals that are not absolutely integrable can have a FT (e.g. sin, cosine, constant, sinc)
  - Signals whose square is absolutely integrable have a Fourier Transform (Plancerel's theorem)

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty \Rightarrow X(f) \text{ exists}$$

Summary

- ❑ Fourier series represents periodic signals as a weighted sum of exponential functions.
  - Square wave has infinite frequency content with FS coefficients following a sinc function
- ❑ Fourier transform represents the spectral components of a aperiodic signal
  - Time-limited signals are not bandlimited and vice versa
  - Stretching a signal along the time axis causes it to shrink along the frequency axis, and vice versa